## Appendix: Derivation of the bound for Q

In the model, the hospital mortality rate is partitioned into two components:

$$M = U + V .$$

Suppose that X denotes the case-mix for a hospital, and let *M(X)* be the expected mortality rate for a hospital with case-mix X: i.e. $M(X) = E(M \mid X)$. The SMR is $\frac{M}{M(X)}$, with variance $\sigma^2_{\mathrm{SMR}}$ , and the proportion of the variation in total mortality explained by case-mix is given by $R^2 = \frac{\mathrm{var}\, M(X)}{\sigma^2_M}$,

with $\sigma^2_M = \mathrm{var}\, M$ . Similarly, $r^2 = \frac{\mathrm{var}\, \mathrm{E}(V \mid X)}{\sigma^2_V}$ is the proportion of the variation in preventable mortality explained by case-mix.

With these notations, the conditional variance formula [1] may be applied to give:

(a) $(1-R^2)\sigma^2_M = \mathrm{E}\,\mathrm{var}(M \mid X)$ and (b) $\sigma^2_{\mathrm{SMR}} = \mathrm{E}\left[M(X)^{-2}\,\mathrm{var}(M \mid X)\right]$

The first assumption mentioned in the text may be formulated as:

A1: For given case-mix, *X*, the components *U* and *V* are conditionally independent.

Under A1 the conditional covariance formula [1] implies

(c) $\mathrm{cov}(SMR, V) = Q\sigma_{SMR}\sigma_V = \mathrm{E}\left[M(X)^{-1}\,\mathrm{var}(V \mid X)\right]$

The second assumption is:

A2: The conditional variances $\mathrm{var}(V \mid X)$ and $\mathrm{var}(M \mid X)$ are constant for all values of the case-mix *X*.

Under (A2), the conditional variances can be taken outside the expectation operators in (a), (b) and (c) leading to:

(e) $Q\sigma_{\mathrm{SMR}}\sigma_V = \mathrm{var}(V \mid X)\mathrm{E}\left[M(X)^{-1}\right] = (1-r^2)\sigma_V^2\,\mathrm{E}\left[M(X)^{-1}\right]$, and (f) $\dfrac{\sigma_{\mathrm{SMR}}^2}{(1-R^2)\sigma_M^2} = \mathrm{E}\left[M(X)^{-2}\right]$.

Furthermore, (g) $(1+t^2)\left[\mathrm{E}M(X)^{-1}\right]^2 = \mathrm{E}\left[M(X)^{-2}\right]$ where $t$ is the coefficient of variation of the quantity $M(X)^{-1}$. It follows from (e), (f) and (g) that $Q = \dfrac{\sigma_V}{\sigma_M\sqrt{1-R^2}}\dfrac{1-r^2}{\sqrt{1+t^2}}$. Therefore

$$Q < \frac{\sigma_V}{\sigma_M\sqrt{1-R^2}} = \frac{\xi c_V}{c_M\sqrt{1-R^2}}, \qquad (2)$$

which is the result used in the paper.

Now replace condition A2 by

A2´: For some constant K, $\mathrm{var}(M \mid X) = KM(X)^2$. Also assume that $\mathrm{var}(V \mid X)$ is non-decreasing as $M(X)$ increases.

The effect of using A2´ in place of A2 is to replace (e) and (f) above by

(e´) $Q\sigma_{\mathrm{SMR}}\sigma_V < (1-r^2)\sigma_V^2\,\mathrm{E}\left[M(X)^{-1}\right]$ and (f´) $\dfrac{\sigma_{\mathrm{SMR}}^2}{(1-R^2)\sigma_M^2} = \left\{\mathrm{E}\left[M(X)^2\right]\right\}^{-1}$.

From which we have $Q < \dfrac{\sigma_V}{\sigma_M\sqrt{1-R^2}}(1-r^2)\mathrm{E}\left[M(X)^{-1}\right]\sqrt{\mathrm{E}\left[M(X)^2\right]}$.

Using a delta technique, $\mathrm{E}\left[M(X)^{-1}\right]$ may be approximated as $\mu^{-1}\left(1+\mu^{-2}\,\mathrm{var}\,M(X)\right) = \mu^{-1}\left(1+R^2c_M^2\right)$ where $\mu = \mathrm{E}M(X) = \mathrm{E}M$ is the mean hospital mortality rate. Similarly, $\sqrt{\mathrm{E}\left[M(X)^2\right]} \approx \mu\left(1+\tfrac{1}{2}R^2c_M^2\right)$. Thus, to leading order in $c_M$, we have the following bound

$$Q < \frac{\sigma_V}{\sigma_M\sqrt{1-R^2}} \times \left(1+\tfrac{3}{2}R^2c_M^2\right)(1-r^2) \qquad (2´)$$

In the base-case ($R^2$ = 0.8, $c_M$ = 0.2) this implies an increase of up to 5% in the previous bound for Q – i.e. in comparison with (2) – and up to 10% in the bound for $Q^2$. However, the increase will be attenuated by the factor $\left(1 - r^2\right)$ and disappears altogether if 5% of the variation in preventable mortality rates can be explained by case-mix (i.e. $r^2$ = 0.05). In any case it is scarcely large enough to disturb the conclusions of the paper.

**Reference**:

1. Ross S. *A First Course in Probability*. 6th Edition. Harlow, UK: Pearson Prentice Hall. 2002.